Delegated Portfolio Management under Ambiguity Aversion

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Abstract

We examine the problem of setting optimal incentives to a portfolio manager (to be employed by an investor through a contract) making an ambiguity-robust portfolio choice with respect to estimation errors in expected returns. We consider a one-period model with a set of risky assets (with multivariate normal returns) whose expected returns are estimated with uncertainty and a linear sharing rule between a risk-neutral investor and a risk averse portfolio manager. The manager accepts the contract if the compensation offered is at least as large as a minimum compensation he determines from his minimum acceptable utility level. Adopting a worst-case $\max - \min$ approach we obtain in closed-form the optimal compensation in various cases where the investor and the manager, respectively adopt or relinquish an ambiguity averse attitude. We apply our result to compute the compensation fees for an investment strategy restricted by Socially Responsible rules.

Keywords: Delegated Portfolio Management, ambiguity, robustness, socially responsible investment.

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1 Introduction

The purpose of this paper is to provide a model for the study of a problem of delegated portfolio management when there is uncertainty on the estimates of the expected returns of the assets in the investment set and the agents are averse to such an ambiguity.

We are interested in analyzing the effect of ambiguity aversion in portfolio choices and managerial fees. To this end we assume that a portfolio manager with an exponential utility function is hired by an investor who pays him a fee on the final wealth produced by the selected portfolio strategy. The manager accepts the contract under the condition that his compensation is at least as large as a minimum level that he sets based on his minimum acceptable level of utility.

The problem we are examining belongs to the theory of Delegated Portfolio Management where the optimal form of a contract between investors and managers are investigated. The seminal paper of this theory is due to Bhattacharya and Pfleiderer (1985) [9]. A nice review of the main contributions can be found in Stracca (2006) [29]. For more recent references see also Fabretti and Herzel (2011) [17]. Our approach to the problem integrates an important line of research in robust portfolio selection; see e.g., [2, 5, 11, 12, 13, 14, 16, 18, 19, 30] and references therein. Robust optimization (RO) grew from the need to address data uncertainties which cannot be easily quantified in terms of probability distributions (imprecise or inexact data), and the obligation to meet certain requirements no matter what the realizations of uncertain data (hard constraints) are. Instead of assuming a probability distribution and formulating a stochastic optimization problem robust optimization confines data uncertainties into an uncertainty set, and follows a worst-case approach which takes full responsibility for all occurrences of data within the uncertainty set, an approach akin to the min-max approach of robust control. The RO methodology begins by defining suitable uncertainty sets (ellipsoidal, polyhedral) which can be justified according to the problem context, and proceeds by transforming the min-max problem into a “tractable” (that can be processed by available efficient algorithms) optimization problem. Specifically, in the context of portfolio selection, it is well-known that distribution of expected returns is not known precisely, and that portfolio composition is particularly sensitive to expected return data [7, 8, 10, 22]. Several authors in the references [1, 5, 11, 12, 14, 16, 18, 19, 25, 30] addressed this problem by applying robust optimization techniques to variants of the portfolio selection problems. The common thread is to treat the uncertainty in the returns by the choice of a suitable uncertainty set, and solve the resulting problem numerically to demonstrate the practical merits of a robust portfolio by means of simulation. The choice of uncertainty sets is either motivated by statistical considerations, e.g. a regression analysis of past data returning
confidence intervals, or dictated by the imperatives of ending up with a tractable (meaning convex) optimization problem.

Our setting deviates from the typical premises of RO in that we assume normally distributed returns for risky assets while we allow for ambiguity in the expected return vector. Hence our approach is more suitably akin to robust stochastic optimization; see [12, 13, 23, 24, 26] for references on this subject where uncertainty is on the distribution of returns rather than on moments of a given distribution. In general, past contributions on robust portfolio selection, with the exception of Garlappi et al (2007) [18], rely on numerical solution of optimization problems whereas in this paper we obtain a closed-form robust portfolio selection rule. Our basic setting is as follows. In a one period economy with $n$ risky assets available for investment and following a multivariate return distribution and a risk-less asset, we deal with an investor who is averse to estimation error in the expected return estimates of the risky assets (we call this ambiguity aversion). The investor, unable (or unwilling) to undertake the investment herself, wants to hire a portfolio manager. She offers a contract which is a linear function of the final (random) wealth, and faces the problem of selecting a suitable fee to be paid to the manager (by maximizing ambiguity robust expected final wealth after paying off the manager), which should be sufficiently high to attract the manager. The manager accepts the contract provided that his utility reservation constraint is satisfied. The main objective of our study is to formulate a model that is simple enough to get explicit results but also sufficiently structured to address important issues such as the impact of the investor’s ambiguity aversion. This impact is measurable from the explicit formulae we obtain. We will show how to apply our results to an example with real market data, determining the sharing rules for different cases of investment strategies driven by a socially responsible approach.

The rest of this paper is organized as follows. We define and study the portfolio selection problem of an ambiguity averse agent in Section 2. In Section 3 we examine the problem of selecting an optimal compensation factor for an ambiguity averse risk neutral investor who delegates investment to a manager. The manager accepts the contract if the compensation factor is at least equal to a minimum level that he sets considering his aversion to ambiguity and his minimum acceptable utility level. The determination of these minimum levels is discussed in Section 4. Section 5 provides an application of our results to the case of a socially responsible investment.
2 Portfolio strategies under ambiguity aversion

Here we consider the problem of an ambiguity averse (AA) agent with a Constant Absolut Risk
Averse (CARA) utility with risk aversion $\alpha$, e.g. a negative exponential utility function. There
are $n$ risky assets with return vector $X$ which follows a Gaussian law with mean $Y$ and positive
definite variance-covariance matrix $\Sigma$. There is also a risk-less asset with return $R \geq 1$.

We model ambiguity as uncertainty in the mean $Y$ of the return vector $X$. We assume that
an ambiguity averse agent considers the set

$$U_X^{(\epsilon)} = \{ Y | (Y - \bar{X})^T \Sigma^{-1} (Y - \bar{X}) \leq \epsilon \},$$

that is an ellipsoid centered at $\bar{X}$ (the estimated mean) and radius the square root of $\epsilon$. The
idea is that the decisions of an AA agent are taken by considering the worst case occurrences
of the true mean within the set $U_X^{(\epsilon)}$. Therefore, more conservative choices are taken when the
set is larger, i.e. for greater values of $\epsilon$. For this reason we will refer to the parameter $\epsilon$ as to
the level of ambiguity aversion. This is of course just one of the possible choices for taking into
account the aversion to ambiguity. We cite Fabozzi et al. [15] for a convincing argument for its
adoption: “The coefficient realizations are assumed to be close to the forecasts ($\bar{X}$), but they
may deviate. They are more likely to deviate from their means if their variability (measured
by their standard deviation) is higher, so deviations from the mean are scaled by the inverse
of the covariance matrix of the uncertain coefficients. The parameter $\epsilon$ corresponds to the
overall amount of scaled deviations of the realized returns from the forecasts against which the
investor would like to be protected.” Moreover, as we shall explain in Section 5, under standard
assumptions on the returns, the random variable

$$(Y - \bar{X})^T \Sigma^{-1} (Y - \bar{X})$$

has a well known distribution, and this fact is also convenient to set the values for $\epsilon$ in order to
get the desired likelihood for the true mean to belong to the set $U_X^{(\epsilon)}$. Garlappi et al. [18] show
that this way of taking into account the ambiguity of estimates may also lead to more stable
portfolio strategies, delivering a higher out-of-sample Sharpe ratio compared to the classical
Markowitz portfolios.

Let us define the vector of excess expected return

$$\bar{\mu} = \bar{E}X - R1$$

where $1$ is a $n$-vector of ones, and the quantity

$$H = \bar{\mu}^T \Sigma^{-1} \bar{\mu}.$$
A standard result from Mean-Variance portfolio theory shows that the slope of the Capital Market Line, given by $\sqrt{H}$, is equal to the maximal Sharpe ratio, i.e.,

$$\sqrt{H} = \max_\omega \frac{\bar{\mu}^T \omega}{\sqrt{\omega^T \Sigma \omega}}.$$  

The slope of the Capital Market Line under ambiguity aversion is obtained by solving the optimization problem

$$\max_\omega \min_{Y \in U_X^{(1)}} \frac{\bar{\mu}^T \omega}{\sqrt{\omega^T \Sigma \omega}}.$$  

After solving the inner problem we get

$$\max_\omega \frac{\bar{\mu}^T \omega - \sqrt{\epsilon} \sqrt{\omega^T \Sigma \omega}}{\sqrt{\omega^T \Sigma \omega}} = \sqrt{H} - \sqrt{\epsilon}. \tag{3}$$

Hence, the effect of the introduction of ambiguity in portfolio choices in the form given by (1) is that of decreasing the perceived slope of the Capital Market Line by a term equal to $\sqrt{\epsilon}$. Therefore, an AA agent whose aversion to ambiguity is so high that he does not trust the forecasted value of the Sharpe ratio, that is for $\epsilon > H$, will refrain from investing in the risky assets. From (3) it also follows that the optimal expected return for an AA agent with an ambiguity aversion smaller than $H$ is unbounded, therefore if he is also risk-neutral he does not have an optimal portfolio strategy.

Let $W_0$ be the initial wealth to be invested, then the AA CARA agent selects a portfolio $\omega \in \mathbb{R}^n$ representing investments in the risky assets by solving the following problem

$$\max_\omega \min_{Y \in U_X^{(1)}} \mathbb{E} \left[ -e^{-\alpha \omega^T (X - R1) + W_0 R} \right]. \tag{4}$$

**Proposition 1** Under the hypotheses of normal returns, an AA agent with CARA utility with risk-aversion $\alpha$ and ambiguity set $U_X^{(1)}$ makes the same portfolio choices as an agent who is not ambiguity averse but is risk-averse with a coefficient of risk-aversion

$$\alpha' = \alpha \frac{\sqrt{H}}{\sqrt{H} - \sqrt{\epsilon}},$$

provided that $\epsilon < H$. When $\epsilon \geq H$ the AA agent is equivalent to an agent with infinite risk-aversion.

**Proof:** It is known that, without ambiguity aversion, the manager solves the problem

$$\max_\omega \mathbb{E} \left[ -e^{-\alpha \omega^T (X - R1) + W_0 R} \right], \tag{5}$$

5
whose solution is
\[ \tilde{\omega} = \frac{1}{\alpha} \Sigma^{-1} \hat{\mu}. \] (6)

We will prove that the optimal portfolio strategy for problem (4) is
\[ \omega^* = \begin{cases} \left( \frac{\sqrt{\nu - \sqrt{\epsilon}}}{\alpha \sqrt{H}} \right) \Sigma^{-1} \hat{\mu} & \text{if } H > \epsilon \\ 0 & \text{if } H \leq \epsilon \end{cases} \] (7)

Then the result follows immediately from comparing the two solutions (6) and (7).

To obtain (7) we recall that the expectation in (4) is equal to
\[ -\exp \left( -\alpha \hat{\mu}^T \omega + \frac{\alpha^2}{2} \omega^T \Sigma \omega - \alpha W_0 R \right). \]

As it is well known (see e.g. Garlappi et al. [18]) solving optimization problem (4) is equivalent to maximizing the function
\[ \hat{\mu}^T \omega - \frac{\alpha}{2} \omega^T \Sigma \omega - \sqrt{\epsilon} \omega^T \Sigma \omega \]
which is a concave function of \( \omega \). Under the hypothesis that the problem admits a solution \( \omega^* \) different from zero, first-order conditions
\[ \hat{\mu} - \alpha \Sigma \omega - \sqrt{\epsilon} \omega^T \Sigma \omega \]
are both necessary and sufficient.

Let us define \( \sigma = \sqrt{\omega^T \Sigma \omega} \), then solving for \( \omega \) in the first-order conditions we obtain
\[ \omega^* = \left( \frac{\sigma}{\sqrt{\epsilon + \alpha \sigma}} \right) \Sigma^{-1} \hat{\mu}. \] (8)

Now, substituting \( \omega^* \) into the equation \( \sigma^2 = \omega^* T \Sigma \omega^* \) we obtain the quadratic equation
\[ (ab)^2 \sigma^2 + 2 \sqrt{\epsilon} \alpha \sigma - H + \epsilon = 0 \]
where \( H = \hat{\mu}^T \Sigma^{-1} \hat{\mu} \). This equation has a positive root
\[ \sigma_+ = \frac{-\sqrt{\epsilon} + \sqrt{H}}{\alpha}, \]
if and only if \( H > \epsilon \). Substituting \( \sigma_+ \) for \( \sigma \) in the expression for \( \omega^* \) in (8) we obtain (7) when \( H > \epsilon \).

When \( H \leq \epsilon \), the only possible solution to the problem is \( \omega^* = 0 \). If no positive root exists, no position is taken in the risky assets and all the wealth is invested into the riskless one.

The previous result states that increasing the level of ambiguity aversion increases the risk aversion, and when \( \epsilon \) is greater than \( H \), the agent will not take any position in the risky assets and put all his wealth into the riskless one.
3 Delegated portfolio management under ambiguity aversion

Now we consider the problem of an investor who wishes to delegate the management of her wealth to a portfolio manager. We assume that the investor is risk-neutral, while the portfolio manager has a CARA utility with risk-aversion \( \alpha \). The investor and the manager may or may not be averse to ambiguity.

The investor allocates a capital \( W_0 \) to the portfolio manager with the mandate to make a portfolio composed of the assets available in the market, including the risk-less one. At the end of the period, the investor will compensate the manager with a fraction \( b \) of the final value of the portfolio keeping for herself the rest. We assume that the manager sets a minimum level \( b_0 \) for the fraction of the final value of the portfolio to accept the contract. The choice of \( b_0 \) will be discussed in detail in Section 4. Therefore, the problem of the investor is to choose the best value for the compensation factor \( b \) according to her utility, by keeping in mind that the value of \( b \) also affects the portfolio strategy of the manager.

From Proposition 1 it follows that an AA manager is equivalent, in terms of portfolio choices, to an Ambiguity neutral agent with an adjusted risk-aversion. For this reason, we can consider, as a general case, the problem of an AA investor who hires an Ambiguity neutral agent with risk aversion \( \alpha \), that is

\[
\max_{b_0 \leq b \leq 1} \min_{\mathbf{Y} \in U^{(6)}_{\mathbf{X}}} \mathbb{E} \left[ (1 - b)(\bar{\omega}^T (\mathbf{X} - \mathbf{R}) + W_0 \mathbf{R}) \right]
\]

\[
U^{(6)}_{\mathbf{X}} = \left\{ \mathbf{Y} \mid (\mathbf{Y} - \bar{\mathbf{X}})^T \Sigma^{-1} (\mathbf{Y} - \bar{\mathbf{X}}) \leq \delta \right\},
\]

where \( \bar{\omega} \) is given in (6).

**Proposition 2** The optimal compensation factor for an AA investor with ambiguity set \( U^{(6)}_{\mathbf{X}} \) dealing with a manager with CARA utility with risk-aversion \( \alpha \) is

\[
b^* = \begin{cases} 
  b_0 & \text{if } \delta \leq H \\
  \left( \sqrt{H(\sqrt{\delta} - \sqrt{\delta})} \right)_{[b_0, 1]} & \text{if } \delta > H
\end{cases}
\]

where \( (\cdot)_{[b_0, 1]} \) represents projection onto the interval \([b_0, 1]\).

**Proof:** We consider first the minimum in (9) where the expected value is obtained changing \( \mathbf{X} \) with \( \mathbf{Y} \). The Lagrangian function is

\[
\mathcal{L}(\mathbf{Y}, \lambda) = (1 - b)(\bar{\omega}^T (\mathbf{Y} - \mathbf{1} \mathbf{R}) + W_0 \mathbf{R}) - \lambda (\delta - (\mathbf{Y} - \bar{\mathbf{X}})^T \Sigma^{-1} (\mathbf{Y} - \bar{\mathbf{X}})).
\]
The inner minimization problem is a convex optimization problem and satisfies Slater condition, therefore optimality conditions are both necessary and sufficient. The first order condition with respect to $Y$ admits the solution

$$Y^* = \bar{X} - \frac{1 - b}{2\lambda} \Sigma \omega$$

which gives

$$\mathcal{L}(Y^*, \lambda) = (1 - b)(\omega^T(\bar{X} - 1R) + W_0R) - \frac{(1 - b)^2}{4\lambda} \omega^T \Sigma \omega - \lambda \delta.$$  

Differentiating and solving for $\lambda$ gives the solution

$$\lambda^* = \frac{1 - b}{2} \sqrt{\frac{\omega^T \Sigma \omega}{\delta}},$$

which finally transforms the problem (9) into

$$\max_b (1 - b) \left( \omega^T (\bar{X} - 1R) + W_0R - \sqrt{\delta \omega^T \Sigma \omega} \right).$$

Hence we substitute the optimal allocation $\bar{\omega}$ in (6) obtaining the problem

$$\max_b (1 - b) \left( W_0R + \frac{\sqrt{H} - \sqrt{\delta}}{\alpha b} \right).$$

The function above is convex for $H \geq \delta$ and concave for $H < \delta$. Differentiating with respect to $b$ we get

$$-\alpha RW_0 - \frac{\sqrt{H} - \sqrt{\delta}}{\alpha b^2}$$

which is always negative if $\delta \leq H$, while it admits the root

$$\sqrt{\frac{\sqrt{\delta} - \sqrt{H}}{\alpha W_0 R}}$$

if $\delta > H$. □

Note that, as we observed above, when the ambiguity aversion $\delta$ is greater than $H$, the optimal choice for a risk-neutral (and also for a risk-averse) investor would be to refrain from investing into risky assets. Therefore, the case $H < \delta$ is only meaningful when the investor must hire a manager also for investing in the risk-free asset.

**Corollary 1** The optimal compensation factor for an AA investor with ambiguity set $U^{(\delta)}_X$ dealing with a CARA utility manager with risk-aversion $\alpha$ and ambiguity set $U^{(\epsilon)}_X$

$$b^* = \begin{cases} 
\left( \frac{\sqrt{H} - \sqrt{\delta})}{\alpha W_0 R} \right)_{[b_0,1]} & \text{if } \delta \geq H \text{ and } \epsilon < H \\
\delta & \text{if } \epsilon \geq H \\
0 & \text{o.w.}
\end{cases}$$

**Proof:** It is sufficient to put $\alpha'$ in the previous results and make the necessary adjustments. □
4 The reservation constraint

We have assumed that the AA manager accepts the contract only if the investor provides him with a share of the final wealth greater than a minimum level $b_0$ chosen by the manager. The choice of $b_0$ depends on a minimum level of utility $\bar{U}$ that the manager expects to receive after signing the contract. In this section we will determine the relation between $b_0$ and $\bar{U}$ and examine some possible ways to determine $\bar{U}$.

The relation between $b_0$ and $\bar{U}$ is given by the equation

$$\min_{Y \in U_X} E[-e^{-\alpha b_0W(\omega^*)}] = \bar{U},$$

which is equal to

$$-\alpha b_0((\omega^*)'\mu + W_0R - \sqrt{\epsilon(\omega^*)'\Sigma \omega^*)} + \frac{(\alpha b_0)^2}{2} (\omega^*)'\Sigma \omega^* = \log(-\bar{U}),$$

where $\omega^*$ is the optimal strategy selected by the manager

$$\omega^* = \mathbb{1}_{\epsilon < H} \left( \frac{\sqrt{H} - \sqrt{\epsilon}}{\alpha b_0 \sqrt{H}} \right) \Sigma^{-1} \bar{\mu},$$

where $\mathbb{1}_A$ is the characteristic function of the set $A$. Substituting and solving for $b_0$ we obtain

$$b_0 = \frac{1}{\alpha W_0R} \left( - \log(-\bar{U}) - \frac{1}{2} \left( \sqrt{\epsilon} - \sqrt{H} \right)^2 \mathbb{1}_{\epsilon < H} \right). \quad (13)$$

When the level of ambiguity aversion $\epsilon$ is smaller than $H$, the portfolio manager will be willing to invest in risky assets, and hence he reduces the sharing factor required to achieve the level of utility $\bar{U}$.

To determine $\bar{U}$ the manager should compute the utility that he may obtain from other competing job offers. We will consider three cases.

In the first case we assume that the manager has the opportunity to work for another investor, offering him the same amount of money $W_0$ to be invested in a set of assets with the same capital market line $H$ but without uncertainty on the expectations, i.e. without ambiguity. The alternative contract provides a sharing factor $\bar{b}_0$. The utility achieved by the manager in this case would be

$$\bar{U} = -e^{-\frac{H}{2} - \alpha \bar{b}_0W_0R}$$

and hence the relation between the minimum required level $b_0$ and the corresponding level "without ambiguity" is

$$b_0 = \bar{b}_0 + \frac{1}{2\alpha W_0R} \left[ H - \left( \sqrt{H} - \sqrt{\epsilon} \right)^2 \mathbb{1}_{\epsilon < H} \right]. \quad (14)$$
The difference between \( b_0 \) and \( \bar{b}_0 \) may be interpreted as a premium for ambiguity, that is an amount that must be offered to the manager to convince him to invest the money in a set of assets where there is uncertainty on the expected returns. Such a premium is always positive and increases with the level of ambiguity aversion \( \epsilon \) because the manager demands an higher premium to invest in more ambiguous assets. The highest premium is reached for \( \epsilon > H \) when the manager will put all the money in the risk-less asset.

In the second case we assume that the alternative contract offers to the manager a share \( \bar{b}_0 \) and the possibility to invest in a set with a slope of the capital market line \( \hat{H} \) without ambiguity on the returns. In this case the manager’s reservation utility is

\[
\bar{U} = -e^{-\frac{\bar{b}_0}{2} - \alpha b_0 W_0 R}
\]

and the minimum fee required becomes

\[
b_0 = \bar{b}_0 + \frac{1}{2 \alpha W_0 R} \left[ \hat{H} - \left( \sqrt{\hat{H}} - \sqrt{\epsilon} \right)^2 1_{\epsilon < \hat{H}} \right].
\]

In this case, when the level of ambiguity aversion is small enough for the agent to invest in the risky assets, there is a term proportional to the difference between the two slopes of the capital market lines \( H \) and \( \hat{H} \). Such a term, that can be positive or negative, represents the premium required by the manager for changing the investment set. Note that this case reduces to the previous one when \( H \) is equal to \( \hat{H} \).

In the final case we consider the situation where the ambiguity is also affecting the alternative investment set, with a level of ambiguity aversion given by \( \rho \). In this case the reservation utility is

\[
\bar{U} = -e^{-\alpha \bar{b}_0 W_0 R - \frac{\left( \sqrt{\bar{H}} - \sqrt{\rho} \right)^2}{2} 1_{\rho < \bar{H}}}
\]

and therefore

\[
b_0 = \bar{b}_0 + \frac{1}{2 \alpha W_0 R} \left[ \left( \sqrt{\bar{H}} - \sqrt{\rho} \right)^2 1_{\rho < \bar{H}} - \left( \sqrt{\bar{H}} - \sqrt{\epsilon} \right)^2 1_{\epsilon < \bar{H}} \right].
\]

We see that the sharing factor is increased by an amount that depends on the perceived slope of the capital market line in the alternative investment set, and it is decreased by an amount that depends on the perceived slope of the capital market line in the actual investment set.

5 An application

We consider an investment mandate subject to some restrictions imposed by the investor. In particular we study investment restrictions due to principles related to Social Responsibility. As
a proxy for the choice of a socially responsible investment strategy we chose the FTSE KLD Social Index (henceforth KLD index). KLD index was launched in 1990 to help socially conscious investors to consider social and environmental factors in their investment choices. It is a float-adjusted, market capitalization-weighted, common stock index, formed by 400 companies from the universe of the 3,000 largest U.S. public equities. The index is composed of approximately 90% of large cap companies, 9% mid cap companies chosen for sector diversification, and 1% small cap companies with exemplary social and environmental records. In 2008 it was renamed FTSE KLD 400 Social Index, while on September 1, 2010 the FTSE KLD indices transitioned to the MSCI ESG Indices. We choose the Vice Fund as a proxy for the shunned investments. The Vice Fund invests in companies engaged in the aerospace and defence industries, owners and operators, gaming facilities as well as manufacturers of gaming equipment, manufactures of tobacco products and producers of alcoholic beverages. There is no intersection between the investment sets considered by the two indexes. The proxy for the risk free assets is the one-month Treasury bill. We are equipped with three time series spanning the period from September 2002 (time $t_0$) to September 2012 (time $t_E$). Figure 1 reports the evolution of an investment of 100 in each of the three assets from $t_0$ to $t_E$.

For each month $t$ we computed the sample mean $\bar{X}_t$ and variance $\Sigma_t$ of the monthly excess returns of the two indices on a moving window of length $T = 36$ (therefore $t$ is between $t_0 + T$ and $t_E$). From these quantities, applying formula (2), we computed the ex-ante optimal Sharpe ratios of an investment combining the risk-free asset with all other assets (the "conventional" strategy), or the KLD index only (the "green" strategy), or the Vice index only (the "sin" strategy).
Figure 2: Maximal ex-ante Sharpe ratios for the three strategies “conventional”, “sin” and “green”. The straight lines represent the tolerances to ambiguity for $n = 2$ (the conventional case) and $n = 1$ (the other cases). When the ex-ante Sharpe ratio is below the tolerance line, an ambiguity averse inverse investor chooses only the risk-free asset.

Strategy).

Figure 2 represents the optimal Sharpe ratios of the three strategies. As it is expected, because of the benefits of diversification, the conventional strategy is always above the other two. For most of the period considered, the sin strategy performed better than the green one. The only two periods where the green strategy outperformed the sin strategy were between the end of 2008 and the beginning of 2009, and between March and October 2011. Note also that the only period where the ”conventional” strategy significantly outperformed both the sin and the green strategy is from August 2009 to June 2011.

To address the issue of ambiguity in the estimates of the expected returns on $n$ assets with a sample of length $T$, denoting by $(\bar{Y} - \bar{X})$ the difference between the exact mean and the sample one, we use a well known result by Hotelling (see Johnson and Wichern (1997) [21], page 212), which states that under standard assumptions on the time series of the returns, when $\Sigma$ is the sample covariance matrix, the random variable

$$Z = \frac{T(T - n)}{(T - 1)n} (\bar{Y} - \bar{X})^T \Sigma^{-1} (\bar{Y} - \bar{X})$$

(15)

follows an F-distribution with $n$ and $T - n$ degrees of freedom.
Therefore, the level of ambiguity aversion $\varepsilon$ can be obtained as a quantile of $Z$ corresponding to a given probability level $p$ of a sample mean falling inside the ellipsoid. The higher the aversion to ambiguity, the higher the value of $p$. In our example, we set $p = 70\%$, obtaining the values $\varepsilon_2 = 0.0714$ for $n = 2$ and $\varepsilon_1 = 0.0307$ for $n = 1$. Figure 2 also reports two constant lines corresponding to the square roots of these values. The ambiguity averse investor refrains from investing in the risky assets whenever the expected Sharpe ratio of the investment is smaller than the square root of the level of ambiguity aversion. Hence, an ambiguity averse conventional investor did not invest in risky assets from December 2008 to December 2011, except for two short periods around January 2009 and January 2011. Note that, while for the period going from June 2005 to March 2009, the ex-ante difference between the Sharpe ratios of the sin and the green strategies would have been significant also for an ambiguity averse investor, it would have been insignificant between December 2008 and December 2011, because an ambiguity averse investor would have decided not to invest separately in any of the risky assets.

To quantify the effect of ambiguity aversion on managerial compensations, we use equation (14) to compute the quantity

$$\Psi_0 = 2\alpha W_0 R (b_0 - \bar{b}_0) = H - \left( \sqrt{H} - \sqrt{\varepsilon} \right)^2 \mathbb{1}_{\epsilon < H}$$

that is a factor proportional to the increase in the sharing factor required by an ambiguity averse manager to invest in an ambiguous set with identical investment opportunities as an unambiguous investment set. In Figure 3 we considered three cases: a green investment set, where investment is restricted to the KLD index, its complement, and the conventional set. We see that the highest compensation is always required by the conventional strategy, while the smallest one is usually required by the green strategy. In other words, the aversion to ambiguity should have affected less a manager investing in the green asset, because the expected utility loss caused by ambiguity when investing in such assets is smaller than the expected loss when investing in the Vice fund alone or in both funds.

To study the effects of the restrictions of the investment set on managerial compensations, with or without ambiguity, we computed the minimal required sharing factor for the different cases reported in Section 4. We considered as the alternative investment set always the total universe of assets and represented in Figure 4 the quantities

$$\Psi_i = 2\alpha W_0 R (b^i_0 - \bar{b}_0), \quad i = 1, \ldots, 4$$

where the index $i$ refers to the following situations

1. $\Psi_1$. The restricted set is the green investment set. No ambiguity aversion.
Figure 3: Pure Ambiguity Premiums for the cases of the Conventional Investment Set ($\Psi_0$), the green investment set ($\Psi_0^G$), and the sin investment set ($\Psi_0^S$)
2. $\Psi_2$. The restricted set is the sin investment set. No ambiguity aversion.

3. $\Psi_3$. The restricted set is the green investment set. With ambiguity aversion at level $\epsilon_2 = 0.714$ for the total universe and $\epsilon_1 = 0.0307$ for the restricted set.

4. $\Psi_4$. The restricted set is the sin investment set. With ambiguity aversion at level $\epsilon_2 = 0.714$ for the total universe and $\epsilon_1 = 0.0307$ for the restricted set.

We note that $\Psi_1$ and $\Psi_2$ are always greater than or equal to zero, which is expected, since they represent the compensation for the restriction in the investment set without ambiguity. The compensation for restricting to green assets $\Psi_1$ is usually greater than $\Psi_2$, because of the higher expected Sharpe ratio of the Sin stock, documented in Figure 2. When taking into account the ambiguity aversion, the premium decreases and sometimes becomes negative. Negative values show that an ambiguity averse manager prefers to invest in the restricted set rather than in the larger one. This happens when the level of ambiguity aversion is high with respect to the expected Sharpe Ratio. Hence, a negative value for $\Psi$ represents a reduction that the manager is willing to apply on the sharing factor. For a long period starting in January 2008 and going as far as September 2012, the factors $\Psi$ are zeros or negative for both the sin and the green investments, therefore an ambiguity averse manager would have not asked for any extra compensation for restricting the investment set.

References


Figure 4: Premium demanded for restricting the investment set to set $X$, with or without ambiguity aversion. Case $\Psi_1$: $X$ is the green set, no ambiguity aversion. Case $\Psi_2$: $X$ is the sin set, no ambiguity aversion. Case $\Psi_3$: $X$ is the green set, with ambiguity aversion. Case $\Psi_4$: $X$ is the sin set, with ambiguity aversion. The level of ambiguity aversion is obtained by setting a likelihood $p = 70\%$. 


